(1) Let $c > 0$ (always), let $a_n := (1+\frac{c}{n})^n$, $b_n := 1 + \frac{c}{1!} + \frac{c^2}{2!} + \dots + \frac{c^n}{n!}$

Show that $a_n \le b_n \quad \forall n \in \mathbb{N}$, (a_n), (b_n) converge and have the same limit.

(2) Conclude that if $a_n := (1-\frac{c}{n})^n$ and $b_n := 1 + \frac{(-c)}{1!} + \frac{(-c)^2}{2!} + \dots + \frac{(-c)^n}{n!}$

then $|a_n - b_n| \to 0$ as $n \to \infty$

(3) Show that $a_n := (1-\frac{c}{n})^n$ converges, hence show (b_n) converges

(It should be easier to show (b_n) converges first, after you learn about Cauchy criterion and absolute conv $\Rightarrow$ Series conv)

(4) Let (x_n) be a bdd sequence.

$A := \{ v \in \mathbb{R} : \text{At most finitely many } n \text{ s.t. } x_n > v \}$.

$B := \{ v \in \mathbb{R} : \text{At most finitely many } n \text{ s.t. } x_n \ge v \}$

Show that $\inf A = \inf B$ (and both inf exist)

Let $x := \inf A$, Show that x is a limit of some subsequence of (x_n).

x is called $\limsup x_n$.

(outline) Ans:

(3): $(1-\frac{c}{n})^n$ converges iff $(1-\frac{c}{n})^n(1+\frac{c}{n})^n$ converges, because $\lim_n (1+\frac{c}{n})^n \ge 1$ exists.

We show that $(1-\frac{c^2}{n^2})^n = 1 + \sum_{k=1}^{n} \frac{c^{2k}(-1)^k}{k!\,n^k}(1-\frac{1}{n})\dots(1-\frac{k-1}{n})$ converges;

Note $$1 - \sum_{k=1}^{n} \frac{c^{2k}}{k!\,n^k} \le (1-\frac{c^2}{n^2})^n \le 1 + \sum_{k=1}^{n} \frac{c^{2k}}{k!\,n^k}$$

and $$\sum_{k=1}^{n} \frac{c^{2k}}{k!\,n^k} \le \frac{1}{n}\sum_{k=1}^{n} \frac{c^{2k}}{k!} = \frac{1}{n}[b_n(c^2)-1] \to 0 \text{ as } n \to \infty$$

By Squeeze thm, $\lim_n (1-\frac{c^2}{n^2})^n = 1$ #

(4) $B \subset A \overset{\text{(Check)}}{\Rightarrow} \inf A \le \inf B$.

$\boxed{\text{Want: } \inf A \ge \inf B}$, which is an equivalent statement of

(a) $\inf B$ is a lower bound of A (Check)

(b) Every lower bound (l.b) of B is a l.b. of A

We use (a): Let $v \in A$, we want to show that $v \ge \inf B$:

$v \in A \Rightarrow x_n > v$ for at most finitely many n

let $\varepsilon > 0$, since $(x_n \ge v+\varepsilon \Rightarrow x_n > v)$, $x_n \ge v+\varepsilon$ for at most finitely many n

$v+\varepsilon \in B$ $\therefore$ $\inf B \leq v+\varepsilon$ $\forall \varepsilon > 0$ i.e. $\inf B \leq v$ #

x is a limit of some subseq. of (x_n):

Let $\varepsilon > 0$; $x-\varepsilon < \inf A = x$ $\Rightarrow$ $x-\varepsilon \notin A$

i.e. $x-\varepsilon < x_n$ for infinitely many n

In particular, take $\varepsilon = 1$, $\exists\, n_1$ s.t. $x-1 < x_{n_1}$

$\varepsilon = \frac{1}{2}$, $\exists\, n_2$ s.t. $x-\frac{1}{2} < x_{n_2}$ and $n_2 > n_1$

$\varepsilon = \frac{1}{3}$, $\exists\, n_3$ s.t. $x-\frac{1}{3} < x_{n_3}$ and $n_3 > n_2$

$\vdots$

Explicitly, define $n_k := \min\{n \in \mathbb{N} : x_n > x - \frac{1}{k}$ and $n > n_{k-1}\}$.

Let $\varepsilon' > 0$, we have the following By AP: $\exists\, N_0 \in \mathbb{N}$ s.t. $\frac{1}{N_0} < \varepsilon'$

and for $k \geq N_0$, we have $x - x_{n_k} < \frac{1}{k} \leq \frac{1}{N_0} < \varepsilon'$ --- 1

By definition of $\inf A =: x$, $x+\varepsilon'$ is not a l.b.

and in our case (why?) everything that is not a l.b of A is an element of A.

$\therefore$ $x+\varepsilon' < x_n$ for at most finitely many n

Take $N_1 := \max\{n \in \mathbb{N} : x_n > x+\varepsilon'\}$

then for $k \geq N_1 + 1$, we have $n_k \geq k \geq N_1+1$ and $x_{n_k} \leq x+\varepsilon'$ i.e. $-\varepsilon' \leq x - x_{n_k}$ ~ 2

$\therefore$ $\exists\, N := \max\{N_0, N_1+1\}$ s.t. $\forall\, k \geq N$,

$|x - x_{n_k}| \leq \varepsilon$

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